

STATISTICAL DISTRIBUTION OF THE STERN SEQUENCE

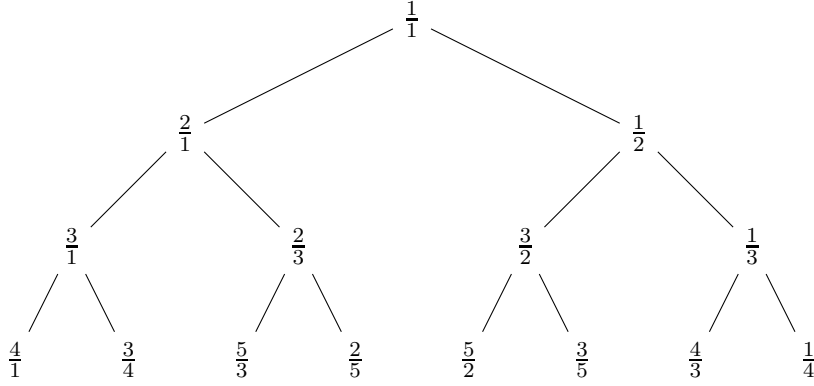
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ABSTRACT. We prove that the Stern diatomic sequence is asymptotically distributed according to a normal law, on a logarithmic scale. This is obtained by studying complex moments, and the analytic properties of a transfer operator.

1. INTRODUCTION

The Calkin-Wilf tree, labelled by the positive rationals [CW00], [AZ14, Chapter 16], is the infinite binary tree starting from the node $\frac{1}{1}$, where a node labelled $\frac{a}{b}$ has children labelled $\frac{a}{b}$ and $\frac{a+b}{b}$, respectively. Each positive rational appears exactly once.

FIGURE 1. First four rows of the Calkin-Wilf tree



The denominators we encounter by this process, reading downwards and from left to right, form a sequence

$$(s(n))_{n \geq 1} = (1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, \dots)$$

called the Stern diatomic sequence [Ste58]. It may alternatively be defined by the particularly simple recurrence relation

$$s(1) = 1, \quad s(2n) = s(n), \quad s(2n+1) = s(n) + s(n+1)$$

for all $n \geq 1$. As such, it is an example of a 2-regular sequence [AS03, Chapter 16, Exercice 32], and enjoys various connections with continued fractions.

Let $\mathcal{I}_N = \mathbb{Z} \cap [2^N, 2^{N+1})$. In this paper, we will study properties of the values $s(n)$ for $n \in \mathcal{I}_N$: these are the denominators of the elements of the $(N+1)$ -th row of the Calkin-Wilf tree described above. Several properties of $(s(n))_{n \in \mathcal{I}_N}$ were recorded by Stern [Ste58] and Lehmer [Leh29]. There has been much interest in understanding the structure of the largest values of $s(n)$ [CT14, Def16, Lan14, Pau16, CS17]: as Lehmer showed, we have $\max_{n \in \mathcal{I}_N} s(n) = F_{N+2}$, where $(F_r)_r$ is the Fibonacci sequence. Recently, Paulin [Pau16] gave a complete description of the $\lfloor N/2 \rfloor$ largest values taken by $(s(n))_{n \in \mathcal{I}_N}$: they are given by various combinations of Fibonacci numbers.

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In the present paper, we are interested in the question of the *statistical* distribution of values of $s(n)$. Our main result is the following estimate for small complex moments of $(s(n))_{n \in \mathcal{I}_N}$.

Theorem 1. *For some $\eta > 0$ and $\theta \in [0, 1)$, there exist holomorphic functions U, V on the disc $\{\tau \in \mathbb{C} : |\tau| \leq \eta\}$ such that*

$$(1.1) \quad \frac{1}{2^N} \sum_{n \in \mathcal{I}_N} s(n)^\tau = \exp \{NU(\tau) + V(\tau)\} (1 + O(\theta^N)).$$

Moreover, $U''(0) \neq 0$.

We deduce from this the following corollary on the value distribution of $(\log s(n))_{n \in \mathcal{I}_N}$.

Corollary 1. *For some constants $\alpha, \sigma > 0$, as N tends to infinity, the values $(\log s(n))$ are asymptotically distributed according to a Gaussian law, with mean αN and variance $\sigma^2 N$: we have, uniformly for $a, b \in \mathbb{R}$ with $a < b$,*

$$\frac{1}{2^N} \left| \left\{ n \in \mathcal{I}_N : \frac{\log(s(n)) - \alpha N}{\sigma \sqrt{N}} \in [a, b] \right\} \right| = \int_a^b \frac{e^{-v^2/2} dv}{\sqrt{2\pi}} + O(N^{-\frac{1}{2}}).$$

Moreover, for some constants ν_1, ν_2 , we have

$$(1.2) \quad \frac{1}{2^N} \sum_{n \in \mathcal{I}_N} \log s(n) = \alpha N + \nu_1 + O(\theta^N),$$

$$(1.3) \quad \frac{1}{2^N} \sum_{n \in \mathcal{I}_N} (\log s(n))^2 = \alpha^2 N^2 + (\sigma^2 + 2\alpha\nu_1)N + \nu_2 + O(N\theta^N).$$

This answers a question of Lansing [Lan13]. More generally, our estimate (1.1) implies asymptotic formulae for $2^{-N} \sum_{n \in \mathcal{I}_N} (\log s(n))^k$, with error term $O_k(N^{k-1}\theta^N)$, for any fixed $k \geq 3$ (see [BV05, formula (2.12)]).

The mean value. Formula (1.2) was recently and independently obtained by Bacher [Bac17, Theorem 12.1]. The constant α appears in our proof with the expression

$$(1.4) \quad \alpha = -\frac{1}{2} \int_0^1 (\log x) d\mu(x),$$

where $d\mu(x) = d?(x)$ is a measure whose distribution function is the Minkowski question mark function [Min05, Fig.7]

$$?([0; a_1, a_2, \dots]) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_1 + \dots + a_n}} \quad \left([0; a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} \right).$$

In [Bac17], Bacher arrives to a different expression for α ,

$$(1.5) \quad \alpha = \log 2 - \sum_{k=1}^{\infty} \frac{m_k}{k 2^k}$$

where $m_k := \int_0^1 x^k d\mu(x)$. Bacher showed that the moments m_k can be computed very accurately, allowing him to obtain a 50-digits approximation for $\alpha \approx 0.39621256\dots$. As mentioned in [Bac17], the constant α is related to work of Kinney [Kin60] on the growth of $?(x)$.

Finally, we remark that the expectation (1.2) can be interpreted as the computation of a relative entropy of the system $([0, 1], \mathcal{F}, \mu)$ given by the Farey map $\mathcal{F}(x) = \min\{\frac{x}{1-x}, \frac{1-x}{x}\}$. This computation leads to a third expression for α , expressing it as

$$(1.6) \quad \alpha = \frac{1}{2} \int_0^1 \log |\mathcal{F}'(x)| d\mu(x).$$

This is again related to a comment made in [Kin60, page 794]. We refer to Section 3.5 below for the details.

Outline of the proof. Our proof of Theorem 1 goes in two main steps. The first step, Lemma 3 below, expresses the moment on the left-hand side of (1.1) in terms of denominators of fractions, whose continued fractions coefficients sum to $N + 1$. The second step is to estimate the distribution of these denominators, by using the connection with the underlying dynamical system given by the Euclid algorithm. To this end we make use of very effective methods from spectral theory of transfer operators [Val03, Rue94]; indeed our situation bears great similarity to work of Baladi-Vallée [BV05], building on earlier work of Hensley [Hen94], on the Gaussian behaviour of Euclidean algorithms.

In Section 2, we express the moments generating function in terms of a certain operator $\mathbb{H}_{\tau,z}$ acting on functions on $[0, 1]$. In Section 3, we describe the analytic properties we will require of $\mathbb{H}_{\tau,z}$. In Section 4, we use the analytic properties of $\mathbb{H}_{\tau,z}$ together with the Cauchy formula to prove Theorem 1 and its corollaries.

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2. EXPRESSING THE MOMENTS GENERATING FUNCTION

For $(\tau, z) \in \mathbb{C}$ and $N \geq 0$, let

$$(2.1) \quad M_N(\tau) := \sum_{n \in \mathcal{I}_N} s(n)^{-\tau}, \quad \mathcal{F}(\tau, z) := \sum_{N \geq 0} z^N M_N(\tau).$$

Since, by [Leh29, Theorem 4], we have $s(n) \ll \phi^N$ for $n \in \mathcal{I}_N$, where ϕ is the golden ratio, the power series $\mathcal{F}(\tau, z)$ converges absolutely in the region

$$\mathcal{D} = \{(\tau, z) \in \mathbb{C} : |z| < (2\phi^{\max\{0, -\Re \tau\}})^{-1}\}.$$

The aim of this section is to reinterpret, in an analytically tractable way, the moments generating function $\mathcal{F}(\tau, z)$. Although we shall not present them so, the results of this section can be interpreted from an ergodic point of view; we discuss this further at (3.13) below.

2.1. Using the 2-regularity. Define the matrices

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For $r \geq 1$, let $\mathcal{M}_r = \{A_{b_1} \cdots A_{b_r}, \forall j, b_j \in \{0, 1\}\}$. We let $\mathcal{M}_0 = \{\text{Id}\}$ and $\mathcal{M} = \cup_{r \geq 0} \mathcal{M}_r$. The (\mathcal{M}_r) form a partition of \mathcal{M} , so that for any $A \in \mathcal{M}$, we may define $|A| = r$ where r is such that $A \in \mathcal{M}_r$. Finally, for a 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $j(A) := c + d$ denote the sum of lower entries of A .

Lemma 1. *For all $N \geq 0$ and $\tau \in \mathbb{C}$, we have*

$$M_N(\tau) = \sum_{\substack{A \in \mathcal{M} \\ |A| = N}} j(A)^{-\tau}.$$

Proof. If $n \in \mathcal{I}_N$, say, $n = 2^N + \sum_{j=0}^{N-1} b_j 2^j$ with $b_j \in \{0, 1\}$, then it is easily seen by induction that

$$\begin{pmatrix} s(n+1) \\ s(n) \end{pmatrix} = A_{b_0} \cdots A_{b_{N-1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and so $j(A_{b_0} \cdots A_{b_{N-1}}) = s(n)$. Taking all possible choices of $(b_j)_{j=0}^{N-1}$ proves our claim. \square

2.2. Expression as denominators of continued fractions. For $\ell \geq 0$, let $\mathcal{A}_\ell := (\mathbb{N}^*)^\ell$. For any sequence $(a_j)_{j=1}^\ell \in \mathcal{A}_\ell$, we define

$$q_{-1}(\mathbf{a}) = 0, \quad q_0(\mathbf{a}) = 1, \quad q_j(\mathbf{a}) = a_j q_{j-1}(\mathbf{a}) + q_{j-2}(\mathbf{a}) \quad (\text{for } 1 \leq j \leq \ell).$$

We also let $\sigma(\mathbf{a}) := \sum_{1 \leq j \leq \ell} a_j$.

We have the following classical lemma (see *e.g.* [FVV]), which is easily proved by induction.

Lemma 2. For $\ell, a_0 \in \mathbb{N}$, and $\mathbf{a} = (a_j)_{j=1}^\ell \in \mathcal{A}_\ell$, we have

$$A_0^{a_0} A_1^{a_1} A_0^{a_2} \cdots A_0^{a_\ell} = \begin{pmatrix} * & * \\ q_{\ell-1}(\mathbf{a}) & q_\ell(\mathbf{a}) \end{pmatrix} \quad (\ell \text{ even}),$$

$$A_0^{a_0} A_1^{a_1} A_0^{a_2} \cdots A_1^{a_\ell} = \begin{pmatrix} * & * \\ q_\ell(\mathbf{a}) & q_{\ell-1}(\mathbf{a}) \end{pmatrix} \quad (\ell \text{ odd}).$$

In particular, in both cases, we have $j(A) = q_{\ell-1}(\mathbf{a}) + q_\ell(\mathbf{a})$ where A denotes the product of matrices on either left-hand sides.

From Lemmas 1 and 2, we deduce the following expression for the moments generating function.

Lemma 3. For $(\tau, z) \in \mathcal{D}$, we have

$$\mathcal{F}(\tau, z) = \frac{1}{1-z} \left(1 + \sum_{\ell \geq 1} \sum_{\mathbf{a} \in \mathcal{A}_\ell} (q_{\ell-1}(\mathbf{a}) + q_\ell(\mathbf{a}))^{-\tau} z^{\sigma(\mathbf{a})} \right).$$

Proof. We have

$$\mathcal{F}(\tau, z) = \sum_{N \geq 0} z^N \sum_{\substack{A \in \mathcal{M} \\ |A| = N}} j(A)^{-\tau} = \sum_{a_0 \geq 0} z^{a_0} \left(1 + \sum_{\ell \geq 1} \sum_{\mathbf{a} \in \mathcal{A}_\ell} (q_{\ell-1}(\mathbf{a}) + q_\ell(\mathbf{a}))^{-\tau} z^{\sigma(\mathbf{a})} \right).$$

and the Lemma follows by summing the geometric series. \square

2.3. Expression as iterates of the transfer operator. For f a function defined on $[0, 1]$, we define the operator

$$(2.2) \quad \mathbb{H}_{\tau, z}[f] : t \mapsto \sum_{m \geq 1} \frac{z^m}{(m+t)^\tau} f\left(\frac{1}{m+t}\right)$$

whenever the right-hand side is well-defined, which is the case when f is bounded and $|z| < 1$. The definition of this operator is inspired from the thermodynamic formalism of the Gauss map; we refer to [Rue02], and chapter 1.2 of [Dod17], for more explanations. We describe the iterates of $\mathbb{H}_{\tau, z}$ on the constant function $\mathbf{1}$. The following is easily proved by induction.

Lemma 4. For $\ell \geq 1$, $|z| < 1$ and $t \in [0, 1]$, we have

$$\mathbb{H}_{\tau, z}^\ell[\mathbf{1}](t) = \sum_{\mathbf{a} \in \mathcal{A}_\ell} (q_\ell(\mathbf{a}) + t q_{\ell-1}(\mathbf{a}))^{-\tau} z^{\sigma(\mathbf{a})}.$$

In particular, summing over $\ell \geq 1$, we obtain

Corollary 2. For $(\tau, z) \in \mathcal{D}$, we have

$$\mathcal{F}(\tau, z) = \frac{1}{1-z} \sum_{\ell \geq 0} \mathbb{H}_{\tau, z}^\ell[\mathbf{1}](1) = \frac{1}{1-z} (\text{Id} - \mathbb{H}_{\tau, z})^{-1}[\mathbf{1}](1).$$

In the next section, we will establish the analytical properties required of the function on the right-hand side.

3. PROPERTIES OF THE TRANSFER OPERATOR

The operator $\mathbb{H}_{\tau,z}$ has been defined and studied at many occurrences in the literature: see [Dod17], [PS92, formula (10)], [Iso02, formula (3.39)]. Indeed, many of the forthcoming properties of $\mathbb{H}_{\tau,z}$ for real z can be found in [PS92]. Nonetheless, to make the arguments as clear as possible for readers unacquainted with the thermodynamic formalism, we will follow [Mor15] and provide full proofs, apart from perturbation theory of operators, which we will quote from [Kat95].

3.1. Definitions. Let $\mathbb{D} = \{t \in \mathbb{C} : |t - \frac{2}{3}| < 1\}$. We will be interested in the spectral properties of the operator $\mathbb{H}_{\tau,z}$ defined at (2.2), acting on the set of functions

$$H^\infty(\mathbb{D}) = \{f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ is holomorphic and bounded}\}.$$

For $f \in H^\infty(\mathbb{D})$, we define $\mathbb{H}_{\tau,z}$ as in (2.2), taking the principal determination of the logarithm.

For \mathcal{T} an operator, we denote by $\text{srd}(\mathcal{T})$ the spectral radius of \mathcal{T} . We further let

$$(3.1) \quad T(x) := \left\{ \frac{1}{x} \right\} \quad (x \in (0, 1]),$$

$$(3.2) \quad T_m(x) := \frac{1}{m+x} \quad (x \in [0, 1]),$$

The Minkowski question mark function is defined in terms of the continued fraction expansion $[0; a_1, a_2, \dots]$ of $x \in [0, 1]$ as

$$(3.3) \quad ?(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_1 + \dots + a_n}}$$

where the sum is finite if $x \in \mathbb{Q}$. It is known that $x \mapsto ?(x)$ is a strictly increasing function [Sal43, page 436]. We introduce the measure $\mu = d?$ on $[0, 1]$. We state and prove here a property which will be needed in the subsequent sections.

Lemma 5. *For all $f \in L^1(\mu)$, we have $\mathbb{H}_{0, \frac{1}{2}}[f] \in L^1(\mu)$ and $\int \mathbb{H}_{0, \frac{1}{2}}[f] d\mu = \int f d\mu$.*

Proof. From the definition (3.3), we have $?(T_m(x)) = 2^{-m}(2 - ?(x))$ for all $m \in \mathbb{N}^*$ and $x \in [0, 1]$. This readily implies that for any interval $A = [\frac{1}{m+1}, \frac{1}{m+x}]$, with $x \in [0, 1]$, we have $\mu(A) = 2^{-m}\mu(T_m^{-1}(A))$. By the monotone class lemma, we obtain $\mu(A \cap [\frac{1}{m+1}, \frac{1}{m}]) = 2^{-m}\mu(T_m^{-1}(A))$ for all μ -measurable set A , and finally $2^{-m} \int_{[0,1]} (f \circ T_m) d\mu = \int_{[1/(m+1), 1/m]} f d\mu$ for all $f \in L^1(\mu)$. Summing over $m \geq 1$ yields our claim. \square

3.2. Decomposition of the transfer operator. We now adapt the arguments of [Mor15] to prove the basic properties of the operator $\mathbb{H}_{\tau,z}$.

Proposition 1. *For all small enough $c_1 > 0$, we may find $c_2 > 0$ and $\theta \in [0, 1)$ such that the following holds. Let*

$$\mathcal{V}_1 := \{(\tau, z) \in \mathbb{C}^2 : |\tau| \leq c_1, |z - \frac{1}{2}| \leq c_1\}.$$

$$\mathcal{V}_2 := \{(\tau, z) \in \mathbb{C}^2 : |\tau| \leq c_2, |z| \leq \frac{1}{2} + c_2, |z - \frac{1}{2}| \geq c_1\},$$

(i) *For $|z| < 1$ and $\tau \in \mathbb{C}$, the operator $\mathbb{H}_{\tau,z}$ is compact and depends holomorphically on (τ, z) in the sense of [Kat95, p.366].*

(ii) *For $(\tau, z) \in \mathcal{V}_1$, we have the decomposition*

$$(3.4) \quad \mathbb{H}_{\tau,z} = \lambda(\tau, z) \mathbb{P}_{\tau,z} + \mathbb{N}_{\tau,z},$$

where $\lambda(\tau, z) \in \mathbb{C}$, and $\mathbb{P}_{\tau,z}$ and $\mathbb{N}_{\tau,z}$ are two compact operators on $H^\infty(\mathbb{D})$. Moreover, the image of $\mathbb{P}_{\tau,z}$ is one-dimensional: $\text{Im } \mathbb{P}_{\tau,z} = f_{\tau,z} \mathbb{C}$ with $f_{\tau,z} = \mathbb{P}_{\tau,z}[\mathbf{1}] \in H^\infty(\mathbb{D}) \setminus \{0\}$.

We have $\text{srd}(\mathbb{N}_{\tau,z}) \leq \theta$, $\mathbb{P}_{\tau,z} \mathbb{N}_{\tau,z} = \mathbb{N}_{\tau,z} \mathbb{P}_{\tau,z} = \mathbf{0}$, and $\lambda(0, \frac{1}{2}) = 1$, $f_{0, \frac{1}{2}} = \mathbf{1}$.

(iii) *For $(\tau, z) \in \mathcal{V}_1$, $f_{\tau,z}$ and $\lambda(\tau, z)$ depend holomorphically on both variables.*

(iv) *For $(\tau, z) \in \mathcal{V}_2$, we have $\text{srd}(\mathbb{H}_{\tau,z}) \leq \theta$.*

Proof. (i) Let $\tilde{\mathbb{D}} := \{t \in \mathbb{C} : |t - \frac{2}{3}| < \frac{17}{16}\}$. We have that T_m maps $\tilde{\mathbb{D}}$ into \mathbb{D} for all $m \geq 1$. In particular, all the elements of the sets $S := \{f \circ T_m \mid f \in H^\infty(\mathbb{D}), \|f\|_\infty \leq 1\}$ can be extended to functions in the unit disk of $H^\infty(\tilde{\mathbb{D}})$. By Montel's theorem the set obtained by these extended functions is pre-compact with respect to the compact-open topology on $H^\infty(\tilde{\mathbb{D}})$ and thus S is pre-compact with respect to the uniform topology of $H^\infty(\mathbb{D})$. In particular, the operator $f \mapsto f \circ T_m$ on $H^\infty(\mathbb{D})$ is compact and thus $f \mapsto \frac{z^m}{(m+.)^\tau} f\left(\frac{1}{m+.}\right)$ is also compact (and holomorphic in (τ, z)). The same then holds for $\mathbb{H}_{\tau,z}$ since its defining series converges locally uniformly in (τ, z) .

(ii) Since $\mathbb{H}_{0,\frac{1}{2}}$ is a compact operator, the non-zero elements of its spectrum are isolated eigenvalues of finite multiplicity; also, notice that $\mathbb{H}_{0,\frac{1}{2}}[\mathbf{1}] = \mathbf{1}$. Let $f \in H^\infty(\mathbb{D})$ be an eigenfunction with eigenvalue λ normalized so that $\max_{t \in [0,1]} |f(t)| = |f(t_0)| = 1$ for some $t_0 \in [0, 1]$. Then, by the definition of $\mathbb{H}_{0,\frac{1}{2}}[f]$ and the triangle inequality we have

$$|\lambda| = |\mathbb{H}_{0,\frac{1}{2}}[f](t_0)| = \left| \sum_{m \geq 1} 2^{-m} f\left(\frac{1}{m+t_0}\right) \right| \leq \sum_{m \geq 1} 2^{-m} = 1.$$

Thus, $|\lambda| \leq 1$. Also, if $|\lambda| = 1$ then the equality holds everywhere and so there exists $c \in \mathbb{C}$ of modulus 1 such that $f\left(\frac{1}{m+t_0}\right) = c$ for all $m \geq 1$. Since f is holomorphic and $\frac{1}{m+t_0}$ has 0 in \mathbb{D} as an accumulation point, we must have $f = c\mathbf{1}$. Thus, $\lambda = 1$ is the only eigenvalue of modulus 1 and $\ker(\mathbb{H}_{0,\frac{1}{2}} - \text{Id})$ is 1-dimensional. Moreover, if $(\mathbb{H}_{0,\frac{1}{2}} - \text{Id})^2[f] = \mathbf{0}$ then $(\mathbb{H}_{0,\frac{1}{2}} - \text{Id})[f]$ is an eigenfunction of $\mathbb{H}_{0,\frac{1}{2}}$ with eigenvalue 1 and so $(\mathbb{H}_{0,\frac{1}{2}} - \text{Id})[f] = c\mathbf{1}$ for some $c \in \mathbb{C}$. Integrating this equation with respect to $d\mu$ by Lemma 5 we find $c = 0$. Thus, f is itself a multiple of $\mathbf{1}$ and so 1 is a simple isolated eigenvalue of $\mathbb{H}_{0,\frac{1}{2}}$.

Now, let C be a small circle centered at 1 which doesn't enclose any other eigenvalue of $\mathbb{H}_{0,\frac{1}{2}}$ and assume $(\tau, z) \in \mathcal{V}_1$. If c_1 is small enough, then by [Kat95, Thm IV.3.16] we have that C doesn't intersect the spectrum of $\text{srd}(\mathbb{H}_{\tau,z})$. It follows that we can write $\mathbb{H}_{\tau,z}$ as a sum of compact operators $\mathbb{H}_{\tau,z} = \mathbb{P}_{\tau,z} + \mathbb{N}_{\tau,z}$ with $\mathbb{P}_{\tau,z}\mathbb{N}_{\tau,z} = \mathbb{N}_{\tau,z}\mathbb{P}_{\tau,z} = 0$ and $\mathbb{P}_{\tau,z}^2 = \mathbb{P}_{\tau,z}$, where $\mathbb{P}_{\tau,z}$ is the Riesz projection associated to C [Kat95, Thm III.6.17]. Moreover, since 1 is a simple eigenvalue, the image of $\mathbb{P}_{0,\frac{1}{2}}$ is one dimensional [Kat95, pp.180-181] and thus the same holds for $\mathbb{P}_{\tau,z}$ if c_1 is small enough [Kat95, Thm IV.3.16]. Also, the spectrum of $\mathbb{H}_{\tau,z}$ restricted to the image of $\mathbb{P}_{\tau,z}$ consists of a unique eigenvalue $\lambda(\tau, z)$, with $\lambda(0, \frac{1}{2}) = 1$, corresponding to the eigenfunction $f_{\tau,z} := \mathbb{P}_{\tau,z}\mathbf{1}$, whereas the spectrum of $\mathbb{N}_{\tau,z}$ consists of that of $\mathbb{H}_{\tau,z}$ with $\lambda(\tau, z)$ removed [Kat95, Thm IV.3.16]. In particular since $\text{srd}(\mathbb{N}_{0,\frac{1}{2}}) \leq \theta' < 1$ for some $\theta' \in [0, 1)$, then by the upper-semicontinuity of the spectral radius [Kat95, Thm IV.3.16] there exists $\theta \in [0, 1)$ such that $\text{srd}(\mathbb{N}_{\tau,z}) \leq \theta$ for all $(\tau, z) \in \mathcal{V}_1$ with c_1 small enough.

(iii) By [Kat95, Thm VII.1.7] $\mathbb{P}_{\tau,z}$ and $\mathbb{N}_{\tau,z}$ depend homomorphically on (τ, z) and thus so does $f_{\tau,z} = \mathbb{P}_{\tau,z}[\mathbf{1}]$. Moreover, since $\mathbb{P}_{0,\frac{1}{2}}[\mathbf{1}](0) = 1$ we have $f_{\tau,z}(0) \neq 0$ for $(\tau, z) \in \mathcal{V}_1$ and c_1 small enough. Thus the holomorphicity of $\lambda(\tau, z)$ follows since $\mathbb{P}_{\tau,z}[f_{\tau,z}](0) = \lambda(\tau, z)f_{\tau,z}(0)$.

(iv) First, consider the case $\tau = 0$ and $z = \frac{1}{2}e^{2\pi i\phi}$ with $\phi \in [0, 1)$. By the triangle inequality we have $\|\mathbb{H}_{0,z}[f]\|_\infty \leq \|f\|_\infty$ for all $f \in H^\infty(\mathbb{D})$; in particular $\text{srd}(\mathbb{H}_{0,z}) \leq 1$ for $|z| = \frac{1}{2}$. Suppose now $\phi \neq 0$. Then since $\mathbb{H}_{0,z}$ is compact, it has an eigenfunction $f \in H^\infty(\mathbb{D})$ with eigenvalue λ of maximum modulus $\text{srd}(\mathbb{H}_{0,z})$. Up to re-scaling f , we can assume $\max_{t \in [0,1]} |f(t)| = |f(t_0)| = 1$ for some $t_0 \in [0, 1]$, whence

$$|\lambda| = |\mathbb{H}_{0,z}[f](t_0)| = \left| \sum_{m \geq 1} 2^{-m} e^{2\pi i m \phi} f\left(\frac{1}{m+t_0}\right) \right| \leq \sum_{m \geq 1} 2^{-m} = 1.$$

If we had $|\lambda| = 1$, then this would mean equality holds everywhere. This implies that all the summands in the first series have the same argument and the m -summand has modulus 2^{-m} , that is

$$e^{2\pi i m \phi} f\left(\frac{1}{m+t_0}\right) = e^{2\pi i \alpha}$$

for some $\alpha \in [0, 1)$. Letting $m \rightarrow \infty$, we deduce $e^{2\pi i m \phi} \rightarrow f(0)$, which implies $\phi = 0$. We conclude that $\text{srd}(\mathbb{H}_{0,z}) < 1$ whenever $|z| = \frac{1}{2}$ and $z \neq \frac{1}{2}$. Then, by the upper semi-continuity of $\text{srd}(\mathbb{H}_{\tau,z})$ [Kat95, Thm IV.3.16], we have that there exists $0 < c_2 < \frac{1}{2}$ such that for τ, z satisfying

$$\frac{1}{2} - c_2 \leq |z| \leq \frac{1}{2} + c_2, \quad |z - \frac{1}{2}| \geq c_1, \quad |\tau| \leq c_2$$

we also have $\text{srd}(\mathbb{H}_{\tau,z}) \leq \theta < 1$. Finally, assume $|z| \leq \frac{1}{2} - c_2$ and $|\tau| \leq c_2 < -\frac{1}{2} \log_2(1 - 2c_2)$. Then, by the triangle inequality for any $f \in H^\infty(\mathbb{D})$, we have

$$\|\mathbb{H}_{\tau,z}[f]\|_\infty \leq \|f\|_\infty \sum_{m \geq 1} |z|^m (m+1)^{c_2}.$$

Since $m+1 \leq 2^m$, computing the series proves that $\text{srd}(\mathbb{H}_{\tau,z}) < 1$. In particular, $\text{srd}(\mathbb{H}_{\tau,z}) < 1$ for all $(\tau, z) \in \mathcal{V}_2$ and (iv) follows. \square

3.3. Specific properties at $(\tau, z) = (0, \frac{1}{2})$. We have the following properties linking $\mathbb{H}_{0,\frac{1}{2}}$ and μ . We recall that the value $\lambda(0, \frac{1}{2}) = 1$ was proved at Proposition 1.(ii).

Proposition 2. (i) We have $\mathbb{P}_{0,\frac{1}{2}}[f] = (\int f d\mu) \mathbf{1}$ for all $f \in H^\infty(\mathbb{D})$.

(ii) For $f, g \in L^1(\mu)$, we have $\mathbb{H}_{0,\frac{1}{2}}((f \circ T) \times g) = f \times (\mathbb{H}_{0,\frac{1}{2}} g)$ almost-everywhere.

(iii) The derivatives of λ satisfy

$$\frac{\partial}{\partial \tau} \lambda(0, \frac{1}{2}) = \int_{[0,1]} \log d\mu, \quad \frac{\partial}{\partial z} \lambda(0, \frac{1}{2}) = 4.$$

Proof. (i) First we recall that $f_{0,\frac{1}{2}} = \mathbf{1}$ and thus $\text{Im } \mathbb{P}_{0,\frac{1}{2}} = \mathbb{C} \mathbf{1}$. Given $f \in H^\infty(\mathbb{D})$, we have $f \in L^1(\mu)$ and so, from Lemma 5, $\int f d\mu = \int \mathbb{H}_{0,\frac{1}{2}}^k[f] d\mu$ for all $k \in \mathbb{N}$. Letting $k \rightarrow \infty$, from the decomposition (3.4) we have $\mathbb{H}_{0,\frac{1}{2}}^k[f] \rightarrow \mathbb{P}_{0,\frac{1}{2}}[f]$ uniformly on $[0, 1]$, and so $\int \mathbb{H}_{0,\frac{1}{2}}^k[f] d\mu \rightarrow \int \mathbb{P}_{0,\frac{1}{2}}[f] d\mu$. Since $\mathbb{P}_{0,\frac{1}{2}}[f] \in \mathbb{C} \mathbf{1}$ is constant, this proves our claim.

(ii) For all $x \in [0, 1]$ and $m \geq 1$, we have $T(T_m(x)) = x$, and so $((f \circ T) \times g) \circ T_m = f \times (g \circ T_m)$. Summing over $m \geq 1$ against 2^{-m} yields the claimed equality.

(iii) We have, following the notations of Proposition 1, $\lambda(\tau, z) f_{\tau,z} = \mathbb{H}_{\tau,z}[f_{\tau,z}]$ for (τ, z) in a neighbourhood of $(0, \frac{1}{2})$. By uniform convergence, we may differentiate term-wise and obtain

$$\begin{cases} \frac{\partial}{\partial \tau} \lambda(\tau, z) f_{\tau,z} + \lambda(\tau, z) \frac{\partial}{\partial \tau} f_{\tau,z} = \mathbb{H}_{\tau,z}[f_{\tau,z} \log + \frac{\partial}{\partial \tau} f_{\tau,z}], \\ \frac{\partial}{\partial z} \lambda(\tau, z) f_{\tau,z} + \lambda(\tau, z) \frac{\partial}{\partial z} f_{\tau,z} = \sum_{m \geq 1} \frac{m z^{m-1}}{(m + \cdot)^\tau} f_{\tau,z} \circ T_m + \mathbb{H}_{\tau,z}[\frac{\partial}{\partial z} f_{\tau,z}]. \end{cases}$$

We evaluate each line at $(\tau, z) = (0, \frac{1}{2})$ and apply the operator $\mathbb{P}_{0,\frac{1}{2}}$, which amounts to integrating against $d\mu$ by point (i). We obtain the claimed identities $\frac{\partial}{\partial \tau} \lambda(0, \frac{1}{2}) = \int \log d\mu$ and $\frac{\partial}{\partial z} \lambda(0, \frac{1}{2}) = \sum_{m \geq 1} m 2^{1-m} = 4$. \square

3.4. Analysis of the pole. The quantity $\rho(\tau)$ defined by the implicit relation $\lambda(\tau, \rho(\tau)) = 1$ will play an important role. As will appear clearly in Section 4, it is the location of the pole giving the dominant contribution in the estimate (1.1).

Proposition 3. For all small enough $c_1 > 0$, there is a unique function ρ , holomorphic on $\{\tau \in \mathbb{C} : |\tau| \leq c_1\}$, and satisfying $(\tau, \rho(\tau)) \in \mathcal{V}_1$ and $\lambda(\tau, \rho(\tau)) = 1$. We have

$$(3.5) \quad \rho(0) = \frac{1}{2}, \quad \alpha := -(\log \rho)'(0) = -\frac{1}{2} \int_{(0,1]} \log d\mu, \quad (\log \rho)''(0) < 0.$$

The proof that $(\log \rho)''(0) < 0$ is not straightforward. As explained in [BV05, Lemma 7] (see also the proof of (6b), page 343 there), it corresponds to a general result about convexity of pressure functions [Rue94, Section 4.6].

In our case, we will mostly follow [Mor15, Proposition 3.3]. As in [BV05, Lemma 7], the argument is mainly adapted from [Bro96]; additionally, we will provide a simpler way to obtain non-positivity (see (3.9) below).

Proof. The well-definedness and holomorphy of ρ follow from the holomorphy of λ on \mathcal{V}_1 , and the fact that $\frac{\partial}{\partial z}\lambda(0, \frac{1}{2}) \neq 0$. This yields also $\rho(0) = \frac{1}{2}$. The value $(\log \rho')(0)$ is easily obtained by differentiating $\lambda(\tau, \rho(\tau))$.

We now wish to study the second derivative. In order to ease comparison with earlier works, we begin by changing variables. Recall that $\alpha = -(\log \rho)'(0)$, and denote

$$\ell(w) := \lambda(-\alpha^{-1}w, \frac{1}{2}e^w).$$

The function $\ell(w)$ is well-defined and analytic in a neighbourhood of $w = 0$; note that by construction, $\ell'(0) = 0$. We differentiate twice the relation $\lambda(\tau, e^{\log \rho(\tau)}) = 1$ and evaluate at $\tau = 0$. We obtain

$$\begin{aligned} (\log \rho)''(0)\rho(0)\frac{\partial}{\partial z}\lambda(0, \frac{1}{2}) &= -(\alpha^2\rho(0)\frac{\partial}{\partial z}\lambda(0, \frac{1}{2}) + \frac{\partial^2}{\partial \tau^2}\lambda(0, \frac{1}{2}) - \alpha\frac{\partial^2}{\partial \tau \partial z}\lambda(0, \frac{1}{2}) + \frac{\alpha^2}{4}\frac{\partial^2}{\partial z^2}\lambda(0, \frac{1}{2})) \\ &= -\frac{1}{2}\alpha^2\ell''(0). \end{aligned}$$

Therefore, our task is to prove that $\ell''(0) > 0$. Define

$$\psi(x) = \begin{cases} \lfloor 1/x \rfloor - \alpha^{-1}\log(x) & (0 < x \leq 1), \\ 0 & (x = 0). \end{cases}$$

Note that $\psi \in L^2(\mu)$. For w in a neighbourhood of 0, let $\xi_w = f_{-\frac{w}{\alpha}, \frac{1}{2}e^w}$ with the notation of Proposition 1. We abbreviate also $\mathcal{L} := \mathbb{H}_{0, \frac{1}{2}}$. By definition, we have

$$\ell(w)\xi_w = \mathbb{H}_{-\frac{w}{\alpha}, \frac{1}{2}e^w}[\xi_w] = \mathcal{L}[e^{w\psi}\xi_w].$$

By local uniform convergence, we may differentiate the above with respect to w : in this way we obtain

$$(3.6) \quad \ell'(w)\xi_w + \ell(w)\frac{\partial}{\partial w}\xi_w = \mathcal{L}[e^{w\psi}(\psi\xi_w + \frac{\partial}{\partial w}\xi_w)].$$

Let $\chi := [\frac{\partial}{\partial w}\xi_w]_{w=0} \in H^\infty(\mathbb{D})$. Evaluating (3.6) at $w = 0$ and using $\ell(0) = 1$, $\ell'(0) = 0$ and $\xi_0 = \mathbf{1}$ yields

$$(3.7) \quad \chi = \mathcal{L}[\psi + \chi].$$

By virtue of the decomposition (3.4), we have $\mathcal{L}^k[\chi] \rightarrow (\int_{[0,1]} \chi d\mu)\mathbf{1}$ uniformly on $[0, 1]$ as $k \rightarrow \infty$. Moreover, since $\mathcal{L}[\psi] \in H^\infty(\mathbb{D})$, and since $\int_{(0,1]} \psi d\mu = 0$ by construction, we deduce that $\|\mathcal{L}^k[\mathcal{L}[\psi]]\|_\infty = \|\mathbb{N}_{0, \frac{1}{2}}^k[\mathcal{L}[\psi]]\|_\infty = O(\theta^k)$. Iterating the relation (3.7), we obtain

$$\chi = \left(\int_{[0,1]} \chi d\mu \right) \mathbf{1} + \sum_{k=1}^{\infty} \mathcal{L}^k[\psi].$$

The point now is that ψ is real-valued, and so the series on the right-hand side is real-valued on $[0, 1]$. In particular, the function $\chi - \chi \circ T$ is real-valued on $[0, 1]$ as well, where we recall that T was defined at (3.1). We will use this fact shortly.

We differentiate again (3.6) with respect to w . We obtain

$$\ell''(w)\xi_w + 2\ell'(w)\frac{\partial}{\partial w}\xi_w + \ell(w)\frac{\partial^2}{\partial w^2}\xi_w = \mathcal{L}[e^{w\psi}(\psi^2\xi_w + 2\psi\frac{\partial}{\partial w}\xi_w + \frac{\partial^2}{\partial w^2}\xi_w)].$$

Evaluating at $w = 0$ yields

$$(3.8) \quad \ell''(0)\mathbf{1} + [\frac{\partial^2}{\partial w^2}\xi_w]_{w=0} = \mathcal{L}[\psi^2 + 2\psi\chi + (\frac{\partial^2}{\partial w^2}\xi_w)_{w=0}].$$

Note that the function $\psi^2 + 2\psi\chi + (\frac{\partial^2}{\partial w^2}\xi_w)_{w=0}$ is in $L^1([0, 1], \mu)$. We integrate both sides of (3.8) against $d\mu$. Since $\int_{[0,1]} \mathcal{L}[f]d\mu = \int_{[0,1]} fd\mu$ for any $f \in L^1(\mu)$, we obtain

$$\ell''(0) = \int_{[0,1]} (\psi^2 + 2\psi\chi) d\mu.$$

Define now $\widehat{\psi} := \psi + \chi - \chi \circ T$. Note also that $\widehat{\psi}$ is real-valued on $[0, 1]$. Using Lemma 5 and Proposition 2 (ii), we have

$$\begin{aligned} \int_{[0,1]} (\chi \circ T)^2 d\mu &= \int_{[0,1]} \mathcal{L}[\chi \circ T] \chi d\mu = \int_{[0,1]} \chi^2 d\mu, \\ \int_{[0,1]} (\psi + \chi)(\chi \circ T) d\mu &= \int_{[0,1]} \mathcal{L}[\psi + \chi] \chi d\mu = \int_{[0,1]} \chi^2 d\mu. \end{aligned}$$

In the last step, we used (3.7). We deduce that

$$\begin{aligned} \int_{[0,1]} (\widehat{\psi})^2 d\mu &= \int_{[0,1]} (\psi + \chi)^2 d\mu - 2 \int_{[0,1]} (\psi + \chi)(\chi \circ T) d\mu + \int_{[0,1]} (\chi \circ T)^2 d\mu \\ (3.9) \quad &= \int_{[0,1]} (\psi + \chi)^2 d\mu - \int_{[0,1]} \chi^2 d\mu \\ &= \ell''(0). \end{aligned}$$

Now, we note that χ is a bounded function, and that $\psi(x) = \lfloor 1/x \rfloor - \alpha^{-1} \log(x)$ tends to $+\infty$ as $x \rightarrow 0$. We may therefore find $\beta > 0$ such that $|\widehat{\psi}(x)| \geq 1$ for all $x \in (0, \beta)$. We deduce that $\ell''(0) = \int_{[0,1]} (\widehat{\psi})^2 d\mu \geq \mu((0, \beta)) = ?(\beta) > 0$ as required. \square

3.5. Dynamical interpretation of α . We now explain how one may interpret formula (1.2) as the computation of a certain relative entropy. Consider the Farey map \mathcal{F} , and the tent map \mathcal{T} , from $[0, 1]$ to itself, given by

$$\mathcal{F}(x) = \begin{cases} \frac{x}{1-x} & (x \in [0, \frac{1}{2}]), \\ \frac{1-x}{x} & (x \in (\frac{1}{2}, 1]), \end{cases} \quad \mathcal{T}(x) = \begin{cases} 2x & (x \in [0, \frac{1}{2}]), \\ 2 - 2x & (x \in (\frac{1}{2}, 1]). \end{cases}$$

By [Pan08, Proposition 1.1], we have $\mathcal{T} = ? \circ \mathcal{F} \circ ?^{-1}$. Since \mathcal{T} preserves the Lebesgue measure, we deduce that \mathcal{F} preserves the measure μ . Consider the partition

$$[0, 1) = \bigvee_{k=0}^N \mathcal{F}^{-k}[0, 1) = \bigcup_{m=0}^{2^N-1} J_{m,N},$$

where $J_{m,N}$ are consecutive segments; explicitly, we have $J_{m,N} = ?^{-1}([\frac{m}{2^N}, \frac{m+1}{2^N}))$. If $\nu(A)$ denotes the Lebesgue measure of A , then

$$\nu(J_{m,N}) = ?^{-1}\left(\frac{m+1}{2^N}\right) - ?^{-1}\left(\frac{m}{2^N}\right), \quad \mu(J_{m,N}) = 2^{-N}.$$

Now, using Proposition 2.1 of [Bac17], we see that

$$(3.10) \quad \nu(J_{m,N}) = \frac{s(m)}{s(2^N + m)} - \frac{s(m+1)}{s(2^N + m+1)} = \frac{1}{s(2^N + m)s(2^N + m+1)}.$$

The second equality stems from the fact that two consecutive fractions of the level N Farey sequence satisfy the determinant condition $|s(m)s(2^N + m+1) - s(m+1)s(2^N + m)| = 1$; it can be checked directly by induction. From the above and since $s(2^{N+1}) = s(2^N)$, we deduce

$$(3.11) \quad \frac{1}{2^N} \sum_{m=0}^{2^N-1} \log s(2^N + m) = -\frac{1}{2} \sum_{m=0}^{2^N-1} \mu(J_{m,N}) \log \nu(J_{m,N}).$$

The right-hand side can now be interpreted as a relative entropy with respect to the partition $(J_{m,N})_m$. As N tends to infinity, a suitable version of Rokhlin's formula [PY98, Theorem 12.10] reads

$$(3.12) \quad -\frac{1}{2^N} \sum_{m=0}^{2^N-1} \mu(J_{m,N}) \log \nu(J_{m,N}) \longrightarrow \frac{1}{2} \int_0^1 \log |\mathcal{F}'(x)| d\mu(x),$$

This is a qualitative version of (1.2), up to identification of the right-hand side with α . Note that our case does not follow directly from [PY98, Theorem 12.10] due the indifferent fixed

point at 0; however, similarly to [PSY98], a proof follows by considering the induced Gauss map [Iso02, Section 2.1]. Also, a stronger statement would follow from the Shannon-McMillan-Breiman theorem [PY98, Theorem 12.11]. We will not detail this here, since this goes beyond our aim.

Let us now show that the quantity on the right-hand side of (3.12) corresponds to our definition (1.4), which amounts to proving the equality (1.6). First we give an alternate expression for α .

Lemma 6. *For the quantity α defined in (3.5), we have*

$$\alpha = \int_0^1 \log(1+x) d\mu(x).$$

Proof. Recall that $\mathcal{L} := \mathbb{H}_{0, \frac{1}{2}}$. We remark that

$$\begin{aligned} \log(1+x) + \mathcal{L}[t \mapsto \log(1+t)](x) &= \log(1+x) + \sum_{m \geq 1} 2^{-m} \log\left(\frac{1+m+x}{m+x}\right) \\ &= \sum_{m \geq 1} 2^{-m} \log(m+x) \\ &= -\mathcal{L}[\log](x) \end{aligned}$$

by splitting the logarithm into a difference. Integrating against $d\mu(x)$ and using Lemma 5 yields $2 \int_0^1 \log(1+x) d\mu(x) = -\int \log d\mu$ as claimed. \square

Since $\log |\mathcal{F}'(x)| = -2 \log(\max\{x, 1-x\})$ and $?(1-x) = 1 - ?(x)$, we deduce that

$$\frac{1}{2} \int \log |\mathcal{F}'(x)| d\mu(x) = -2 \int_{1/2}^1 (\log x) d\mu(x) = \int_0^1 \log(1+x) d\mu(x)$$

where we have used the equality $\mathcal{L}[t \mapsto \mathbf{1}_{t > 1/2} \log t](x) = -\frac{1}{2} \log(1+x)$. By Lemma 6, the claimed equality (1.6) follows.

Remark. In Theorem 12.1 of [Bac17], an alternate definition is given for the constant α . To see that it coincides with ours, we write, using Lemma 6,

$$\alpha = \int_0^1 \log(1+x) d\mu(x) = \int_0^1 \log(2-x) d\mu(x) = \log 2 + \int_0^1 \log(1 - \frac{x}{2}) d\mu(x)$$

where we have used again $?(1-x) = 1 - ?(x)$. Expanding the logarithm into a power series, we recover the formula for α given in (1.5). Conjectures 8.1 and 8.2 of [Bac17], which are concerned with similar identities, can be proven along the same lines.

Remark. From the ergodic point of view, Lemma 1 above can be seen as the equality

$$\begin{aligned} (3.13) \quad \{s(2^N + m), 0 \leq m \leq 2^N\} &= \{\text{denom}(?(x)), x \in \mathcal{T}^{-N}(\{0, 1\})\} \\ &= \{\text{denom}(x), x \in \mathcal{F}^{-N}(\{0, 1\})\} \end{aligned}$$

where we have used [Bac17, Proposition 2.1] and [Pan08, Proposition 1.1]. Moreover, Lemma 2 corresponds to passing from the Farey system $([0, 1], \mathcal{F}, \mu)$ to the induced system on $[\frac{1}{2}, 1]$, which is given by the Gauss map.

4. PROOF OF THEOREM 1

Recall that the sets \mathcal{V}_1 and \mathcal{V}_2 were defined in Proposition 1. For $(\tau, z) \in \mathcal{V}_2$, we have $\text{srd}(\mathbb{H}_{\tau, z}) \leq \theta < 1$. Therefore, the series

$$\sum_{\ell \geq 0} \mathbb{H}_{\tau, z}^{\ell}[\mathbf{1}](1)$$

converges uniformly and defines an analytic function throughout $z \in \mathcal{V}_2$, and so does $\mathcal{F}(\tau, z)$ by Corollary 2. For $|\tau| \leq c_2$, using the definitions (2.1) and the Cauchy formula, we deduce

$$M_N(\tau) = \frac{1}{2\pi i} \oint_{\gamma_1 \cup \gamma_2} z^{-N} \mathcal{F}(\tau, z) \frac{dz}{z},$$

where γ_1 (resp. γ_2) is the arc $\{z : |z| = \frac{1}{2} + c_2, |z - \frac{1}{2}| \geq c_1\}$ (resp. $\{z : |z - \frac{1}{2}| = c_1, |z| \leq \frac{1}{2} + c_2\}$), run counter-clockwise with respect to the origin.

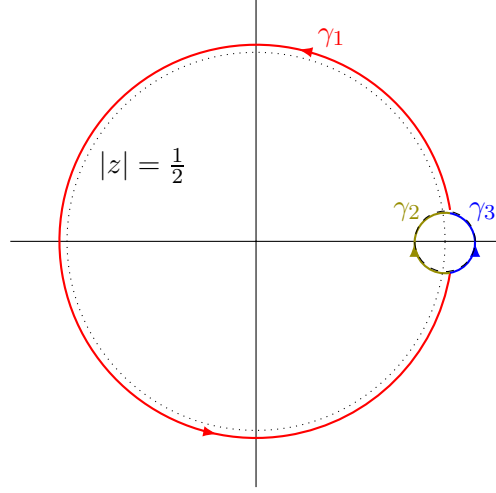


FIGURE 2. The contours γ_1 , γ_2 and γ_3

For $z \in \gamma_1$, we have $(\tau, z) \in \mathcal{V}_2$ and so $\mathcal{F}(\tau, z)$ is bounded uniformly. We deduce

$$(4.1) \quad \frac{1}{2\pi i} \oint_{\gamma_1} z^{-N} \mathcal{F}(\tau, z) \frac{dz}{z} \ll \left(\frac{1}{2} + c_2\right)^{-N} \ll 2^N (1 + 2c_2)^{-N}.$$

This yields an acceptable error term.

For $z \in \gamma_2$, we have $(\tau, z) \in \mathcal{V}_1$ and so we may write, according to the decomposition (3.4),

$$\mathcal{F}(\tau, z) = \frac{\lambda(\tau, z)}{(1-z)(1-\lambda(\tau, z))} \mathbb{P}_{\tau, z}[\mathbf{1}](1) + \frac{1}{z-1} (\text{Id} - \mathbb{N}_{\tau, z})^{-1}[\mathbf{1}](1) \quad (|z - \frac{1}{2}| \leq c_1, z \neq \rho(\tau)).$$

Since $\text{srd}(N_{\tau, z}) \leq \theta$ throughout $|z - \frac{1}{2}| \leq c_1$, the term $(\text{Id} - \mathbb{N}_{\tau, z})^{-1}[\mathbf{1}](1)$ defines an analytic function on $|z - \frac{1}{2}| \leq c_1$. The same fact holds for $\mathbb{P}_{\tau, z}[\mathbf{1}](1)$. Finally, the function $z \mapsto \lambda(\tau, z) - 1$ has a unique zero, which is simple, at $z = \rho(\tau)$. By the residue theorem, we have

$$\frac{1}{2\pi i} \oint_{\gamma_2} z^{-N} \mathcal{F}(\tau, z) \frac{dz}{z} = \frac{R(\tau)}{\rho(\tau)^N} + \frac{1}{2\pi i} \oint_{\gamma_3} z^{-N} \mathcal{F}(\tau, z) \frac{dz}{z},$$

where

$$R(\tau) = \frac{\mathbb{P}_{s, \rho(\tau)}[\mathbf{1}](1)}{\rho(\tau)(1 - \rho(\tau)) \frac{\partial}{\partial z} \lambda(\tau, \rho(\tau))}$$

and γ_3 is the arc $\{z : |z - \frac{1}{2}| = c_1, |z| \geq \frac{1}{2} + c_2\}$. By an argument identical to (4.1), we have

$$\frac{1}{2\pi i} \oint_{\gamma_3} z^{-N} \mathcal{F}(\tau, z) \frac{dz}{z} \ll 2^N (1 + 2c_2)^{-N}.$$

Finally, we note that $R(\tau)$ is analytic for $|\tau| \leq c_2$ and $R(0) = 1$. At the possible cost of reducing c_2 , we may write $R(\tau) = \exp V(\tau)$, and similarly $\rho(\tau) = \frac{1}{2} \exp U(\tau)$, for two functions U and V holomorphic for $|\tau| \leq c_2$. In the same range of τ , we conclude that

$$(4.2) \quad 2^{-N} M_N(\tau) = \exp(NU(\tau) + V(\tau)) \{1 + O(\eta^N)\}$$

for some $\eta \in [0, 1)$. This is the statement of Theorem 1, up to changing τ to $-\tau$. Corollary 1 follows at once using Hwang's quasi-power theorem, *e.g.* [FS09, Lemma IX.1], [Hwa98], [Hwa96].

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